Convex Design Control for Practical Nonlinear Systems

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Abstract—This paper describes a new control scheme for Approximately Optimal Control (AOC) of nonlinear systems, Convex Control Design (ConvCD). The key idea of ConvCD is to transform the approximate optimal control problem into a convex Semi-Definite Programming (SDP) problem. Contrary to the majority of existing AOC designs where the problem that is addressed is to provide a control design which approximates the performance of the optimal controller by increasing the ‘controller complexity’, the proposed approach addresses a different problem: given a controller of ‘fixed complexity’ it provides a control design that renders the controller as close to the optimal as possible and, moreover, the resulted closed-loop system stable. Two numerical examples are used to show the effectiveness of the method.

Index Terms—Approximately Optimal Control (AOC), Control Lyapunov Function (CLF), Convex Control Design (ConvCD), Semi-Definite Programming (SDP)

I. INTRODUCTION

Despite the impressive recent advances in the control design for nonlinear systems, the problem of providing a scalable, practically feasible, optimal controller has not found yet a definite answer even for nonlinear systems of small-scale. The “curse-of-dimensionality”, a notion introduced by Richard Bellman [1] decades ago, still haunts control engineering. With the exception of the narrow class of nonlinear systems that can be transformed into linear ones by means of transformations and/or feedback, computing the optimal controller actions is an NP-complete problem and thus impossible to be implemented in real-time. Despite the significant advances achieved over the last years in formulating optimal nonlinear control theory [2], [3], [4], [5], [6], these methods have not yet progressed into control applications because of the difficulties associated to the solution of the Hamilton-Jacobi partial differential equation.

Model Predictive Control (MPC) for nonlinear systems, a control approach which has been extensively analysed and successfully applied in industrial plants during the latest decades [7], [8], [9], faces similar dimensionality issues. In fact, in most cases, predictive control computations for nonlinear systems amount to numerically solving online a non-convex high-dimensional mathematical programming problem. While for linear and/or constrained linear systems the optimization problem expressing the performance index is a Quadratic Program, or alternatively, a Linear Program, for which efficient numerical routines have been developed, the solution of the optimization problem for nonlinear systems may require a quite formidable computational burden if online solutions are required.

Optimal solutions based on control Lyapunov functions (CLFs) guarantee closed-loop stability whenever a CLF can be found, and do not require the solution of possibly intractable optimization problems [10], [11]; however, no systematic procedure exists for finding CLFs of arbitrary nonlinear systems.

Although computing and realizing optimal controllers for general nonlinear systems is practically infeasible, recently there has been a significant effort and research activity towards designing and implementing Approximately Optimal Controllers (AOC), i.e., providing a control design that is both practically feasible and approximates the performance of the optimal controller at a satisfactory level, [12], [13], [14], [15], [16]. The key idea in all these approaches is to use approximation techniques in order to come up with a practically feasible control design that effectively approximates the optimal controller. However, although the use of approximation techniques significantly reduces the burden involved in the optimal control design, the majority of AOC designs still face the “curse-of-dimensionality” problem as they require the solution of highly non-convex problems. This, in turn, may lead to either impractical designs or optimization algorithms that might get trapped into local minima.

Recently, different AOC designs have been proposed and developed: such designs transform the problem of constructing an AOC into a convex problem [17], [18], [19], [20]. In [20] both the system dynamics and the optimal controller are approximated using PieceWise Linear (PWL) approximators. Using such approximations the optimal control problem is approximated as a piecewise Linear Quadratic (LQ) problem which admits a PieceWise Quadratic (PWQ) Lyapunov (=optimal cost-to-go) function. In [20] the requirement for the PWQ Lyapunov function to be continuous at the switching boundaries, along with the Hamilton-Jacobi-Bellman (HJB) equations, results in an overall design that requires the solution of a convex optimization problem involving the solution of Linear Matrix Inequalities (LMIs).
A different convex AOC design was proposed and analyzed in [17], [18], [19]. This design combines the transformations proposed in [21] for the control design of polynomial nonlinear systems using Sum-of-Squares (SoS) polynomials, with function approximation results and approximation of positive functions by SoS polynomials. The design of [17], [18], [19] involves polynomial approximations for the system dynamics, the controller and the optimal cost-to-go function. The resulting control design requires the solution of a convex problem (a quadratic Semi-Definite Programming–SDP problem which can be transformed into a standard LMI problem using Schur complement techniques). There are several advantages of the approach of [17], [18], [19] as compared to the approach of [20]: (a) the approach of [17], [18], [19] can employ and handle nonlinearities in the controller structure, while the one of [20] is restricted to employing purely linear controllers; (b) it employs smooth controllers, (c) it avoids the requirement for restricting the switching Lyapunov functions to be continuous at the switching boundaries and (d) most importantly, it can approximate with arbitrary accuracy the performance of any – feasible – stabilizing controller for the system. However, the price paid for these advantages is the significant increase of the computational requirements: while the approach in [20] requires the solution of LMIs that are of the same order as the system dimension, the size of LMIs in [17], [18], [19] increases exponentially with the order of the polynomial approximation used.

In this paper, we propose and analyze a new convex control design methodology that combines the advantages of both approaches in [20] and [17], [18], [19]. The main features of this new control design — abbreviated as ConvCD (Convex Control Design) — are:

1) A convex AOC methodology which requires the solution of a convex problem (involving either SDP or LMIs).

2) Like the majority of existing AOC designs, it can approximate with arbitrary accuracy the performance of the respective optimal controller by increasing the “controller complexity” (e.g., by increasing the number of switching linear controllers).

3) The controller complexity must not be “large” for the resulting ConvCD design to provide a stable and efficient performance. This is particularly useful in practical applications where the complexity of an efficient controller is known and given (i.e., the number of switching linear controllers as well as the strategy for switching among the controllers is given). Given such a controller of fixed complexity, ConvCD provides a controller which is as close to the optimal as possible while guaranteeing closed-loop system stability. Tuning parameters can be used to regulate the transient response (in terms of overshoot and exponential decay) and to have an estimate of the “distance” of the resulting controller from the optimal one.

4) The proposed ConvCD methodology overcomes the scalability and computational problems of [17], [18], [19]. In particular, similarly to [20], it involves SDP constraints (or LMIs) of the same order as the system state dimension.

5) It can handle convex state and control constraints.

6) The proposed scheme avoids the requirement of continuity of the Lyapunov function at the switching boundaries. This is made possible by using smooth mixing signals that make sure that the switching among the space partition is performed smoothly.

7) Finally, the proposed approach is able to provide control designs which use PieceWise NonLinear (PWNL) approximation control schemes instead of PWL ones. This property may prove to be very useful in practical applications, where a tedious and complicated design (e.g., for partitioning of the state space, etc.) is required for efficient PWL approximation while the corresponding design for PWNL approximation emerges naturally.

The paper is organized as follows: in Sect. II a simple example is presented to introduce the ConvCD scheme. The problem formulation for the general case is presented in Sect. III. Sect. IV deals with the transformations and approximation in order to put the nonlinear system in a suitable form for the ConvCD design. Sect. V exposes the ConvCD approach, together with the stability results. Sect. VI shows the effectiveness of the proposed method by solving a more complex nonlinear problem.

II. A TUTORIAL EXAMPLE

In this section, we use a simple example to introduce the ConvCD approach. Consider the problem of designing a state feedback controller for the inverted pendulum model of [20]

\[
\begin{align*}
\dot{\chi}_1 &= \chi_2 & (1) \\
\dot{\chi}_2 &= -0.1\chi_2 + \sin(\chi_1) + u & (2)
\end{align*}
\]

where \(\chi = [\chi_1 \, \chi_2]^{T}\) is the state of the pendulum, composed of angular position and angular velocity, and \(u\) is the control input torque. We are interested in applying the ConvCD technique to find a state feedback control that brings the pendulum to the upright unstable equilibrium \((0, 0)\), while minimizing the criterion

\[
J = \int_{0}^{\infty} \left(4\chi_1^{2}(s) + 4\chi_2^{2}(s) + u^2(s)\right) ds \quad (3)
\]

The first step in the proposed approach is to impose special transformations that render the system dynamics and the objective criterion in an appropriate format that is convenient for our developments. In order to keep the example as simple as possible, no constraints will be considered for this example. Input/state constraints will be considered in Section III, where the ConvCD formulation will be exposed for a more general set of control problems.

After the addition of an integrator \(\dot{\bar{u}} = v\) we define the augmented state vector \(x = [\chi^T \, u^T]^T\). The second step in the proposed design is to approximate the system dynamics as well as the objective criterion (3) using smooth mixing signals. For the inverted pendulum case, a PieceWise Linear (PWL) model of (1)-(2) can be constructed by finding a PWL approximation of the system nonlinearity \(\sin(\chi_1)\). Similarly to [20] such nonlinearity is approximated inside the convex
interval $-4 \leq \chi_1 \leq 4$. For reasons that will be clearer as the ConvCD methodology is exposed, contrary to [20], where a switching PWL approximation is considered, here we look for a smooth PWL approximation: this is made possible by using smooth mixing signals $\beta = [\beta_1 \ldots \beta_L]^T$ that make sure that the switching among the space partition is performed smoothly. Besides, by construction, the mixing signals must be non-negative and normalized, i.e., $\sum_{i=1}^{L} \beta_i(\chi_1) = 1$, $\forall \chi_1 \in \mathbb{R}$. The interval $[-4, 4]$ is divided into $L = 5$ overlapping subsets and five mixing signals are constructed on the basis of the Gaussian function $\psi(\chi_1, \chi_0, \sigma_{\chi}) = e^{-\frac{(\chi_1 - \chi_0)^2}{2\sigma_{\chi}^2}}$. Consider the pre-normalized weights, $\beta = [\beta_1 \ldots \beta_L]^T$, where $\beta_i(\chi) = \psi(\chi, \chi_0, \sigma_{\chi})$, and $\chi_0$ is the center of the Gaussian function, which belong to a set of the five centers which have been chosen as $\{-3.2, -1.6, 0, 1.6, 3.2\}$, and $\sigma_{\chi} = 0.2667$ is the bell width. Different widths can be selected for each Gaussian function: in this example the width is the same for each bell, since this choice has lead to satisfying results. In order to obtain a normalized set of mixing signals, the mixing signals $\beta_i(\chi_1)$ are generated by normalizing $\beta$, i.e., $\beta_i(\chi_1) = \beta_i(\chi_1)/\sum_{j=1}^{L} \beta_j(\chi_1)$, $i = 1, \ldots, L$. The mixing signals resulting from this procedure are shown in Fig. 1(a). Once the mixing signals have been defined, it is possible to approximate the $\sin(\chi_1)$ term by:

$$\sin(\chi_1) = \beta_1(\chi_1) \cdot (-\theta_1(\chi_1 - \theta_2) + \beta_2(\chi_1) \cdot (-\theta_1) + \beta_3(\chi_1) \cdot (\theta_1(\chi_1 + \beta_4(\chi_1) \cdot (-\theta_1) + \beta_5(\chi_1) \cdot (-\theta_1(\chi_1 + \theta_2)
$$

(4)

Note that the anti-symmetry of the sin function has been taken into account for the piecewise linear approximation. The vector $[\theta_1 \theta_2 \theta_3 \theta_4]^T$ can be found using a Least Squares method. Fig. 1(b) shows the PWL approximation of $\sin(\chi_1)$ via (4) inside the convex interval $[-4, 4]$. For the inverted pendulum case, the partition (4) can be viewed as a partition in the variable $\chi_1$, while the partition is independent of $\chi_2$. In Sect. III the mixing functions $\beta_i$ will be generalized so as to be functions of the entire state $x$. For this reason, from now on, let us denote the mixing functions by $\beta_i(x)$, $i = 1, \ldots, L$.

After the described PWL approximation, we arrive at a system of the form

$$\dot{x} \approx \sum_{i=1}^{L} \beta_i(x) (A_i \bar{x}(x) + Bu)$$

(6)

where $\bar{x}(x) = [\chi_1 \chi_2 1 u]^T$, $B = [0 \ 0 \ 1]^T$, and $A_i$ depending on the coefficients $\theta$ as shown in (5). The system (6) can be rewritten as

$$\dot{x} \approx \Phi(x)z(x) + Bv$$

(7)

where

$$\Phi(x) = [\sqrt{\beta_1(x)}A_1 \ldots \sqrt{\beta_L(x)}A_L]$$

and the vector $z(x)$ is defined as follows:

$$z(x) = \begin{bmatrix} \sqrt{\beta_1(x)}\bar{x}(x) \\ \vdots \\ \sqrt{\beta_L(x)}\bar{x}(x) \end{bmatrix}$$

(8)

Accordingly, the objective function (3) to be minimized can be written in the form

$$J = \int_{0}^{\infty} \sum_{i=1}^{L} \beta_i(x) (\bar{x}^T Q_i \bar{x}) \ ds$$

(9)

by defining the weight matrices $Q_i = \text{blkdiag} \{4, 4, 0, 1\}$, $i = 1, \ldots, L$.

By applying all the transformations, the optimal state feedback design problem can be cast as

$$\text{minimize } J = \int_{0}^{\infty} (z^T(s)Qz(s)) \ ds$$

subject to

$$\dot{x} = \Phi(x)z(x) + Bv + \nu$$

(11)

where $\nu$ stands for the approximation error due to the replacement of the actual system dynamics (1)-(2) by (7).

Application of the Hamilton-Jacobi-Bellman (HJB) equation to the above problem results in the following equation

$$-z^T Q z = \frac{\partial V^*}{\partial x}(x) (\Phi(x)z(x) + Bv^* + \nu)$$

(12)

where $V$ is the optimal-cost-to-go function, i.e.,

$$V(x(t)) = \int_{t}^{\infty} (z^T(s)Qz(s)) \ ds,$$

$v^*$ denotes the optimal control and $\nu$ is the error approximation term that results from the earlier approximations.

The optimal cost-to-go function $V$ is a Control Lyapunov Function (CLF) for the controlled system: if $\chi_1 \in X$, the CLF $V$ can be approximated – with accuracy $O(1/L)$ – using a piecewise approximation of quadratic Lyapunov functions, as follows:

$$V(x) \approx \sum_{i=1}^{L} \beta_i(x) \bar{x}^T(x) \bar{P}_i \bar{x}(x) = z^T(x) \bar{P} z(x)$$

(13)

where $\bar{P}$ is a constant positive definite matrix with $\bar{P}$ being symmetric and having the following block diagonal form

$$\bar{P} = \begin{bmatrix} P_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & P_L \end{bmatrix}$$

(14)

where $P_i$ are $\dim(\bar{x})^2$-dimensional symmetric and positive definite matrices.
The approximation of the optimal-cost-to-go function via (13) can be shown mathematically also in the general case (e.g., $X \subset \mathbb{R}^n$, with $n > 1$, and $\beta_i$ possibly depending on the entire state $x$) and it will be shown in Sect. V (cf. Lemma 1).

Similar to the use of approximations for the optimal cost-to-go function, the optimal controller $v^*$ can be approximated as follows:

$$v^* \approx \sum_{i=1}^{L} \beta_i(x)G_i(z(x) = \Gamma(x)Gz(x)$$  \hspace{1cm} (15)

where $G_i$ are constant matrices, and

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_L \end{bmatrix}, \quad \Gamma(x) = [\beta_1(x)I, \ldots, \beta_L(x)I]$$

Using the approximations (7), (13) and (15), the HJB Equation (12) can be written as follows:

$$0 = z^T\left(\tilde{\Phi}(x) + B\Gamma(x)G\right)^T M_x^\tau P + PM_z\tilde{\Phi}(x) + B\Gamma(x)G + Q)z - \tilde{\nu}$$  \hspace{1cm} (16)

where $M_z \equiv M_z(x)$ denotes the matrix whose $(i, j)$-th entry is given by $M_z_{i,j}(x) = \partial z(x)/\partial x_i$, and $\tilde{\nu}$ is the approximation error term that is inversely proportional to the number $L$ of mixing signals, resulting from the approximations (7), (13) and (15). Moreover, as the optimal cost-to-go function $V(x)$ is a CLF for the system (7) we have that closed-loop stability is preserved by the control scheme if $\tilde{V} < 0$ for $x \neq 0$, or, equivalently using the approximations (7), (13) and (15) if the following inequality holds

$$\mathcal{L}_{P,G}(x) \triangleq z^T\left(\tilde{\Phi}(x) + B\Gamma(x)G\right)^T M_x^\tau P + PM_z\tilde{\Phi}(x) + B\Gamma(x)G + Q)z < 0, \quad \forall x \not\in B(\tilde{\nu})$$  \hspace{1cm} (17)

Equations (16) and (17) indicate that it suffices to choose $P,G$ so that the term $\mathcal{L}_{P,G}(x)$ is as small as possible subject to the constraint that $\mathcal{L}_{P,G}(x)$ is – almost – negative definite. In other words, the problem of constructing an approximately optimal performance can be cast as the following optimization problem:

$$\min \|\mathcal{G}_{P,G}(x)\|^2 + \gamma$$  \hspace{1cm} (18)

s.t. $P > 0$

$$\mathcal{L}_{P,G}(x) \leq -\gamma, \quad \gamma > c > 0, \forall x \not\in B(\tilde{\nu})$$

where $c$ is a small positive user-defined constant and where $B(\tilde{\nu})$ denotes a ball centered at the origin and having radius proportional to $\tilde{\nu}$.

Unfortunately, as the above optimization problem is nonlinear with respect to the unknowns $P, G$, attempting to solve (18) is non-convex – and thus difficult to solve – even in the case where the approximation-related term $\tilde{\nu}$ is negligible. To circumvent this problem we work similarly to [21], [17]-[19]: by multiplying the terms inside the parenthesis of (16) by $P^{-1}$ from left and right we obtain that

$$\mathcal{F}_{P,F,Q}(x) \triangleq z^T\left(\left[\tilde{\Phi}(x)\right]^T + F^\tau \Gamma(x)B\right)^T M_x^\tau + M_z\tilde{\Phi}(x)P + B\Gamma(x)F + Q)z$$

$$= \tilde{\nu}$$  \hspace{1cm} (19)

where $\tilde{\nu} = \mathcal{O}(\tilde{\nu})$,

$$P \triangleq P^{-1}, \quad Q \triangleq PQP \equiv P^{-1}Q P^{-1}$$  \hspace{1cm} (20)

and $F$ is a matrix satisfying

$$F = GP$$  \hspace{1cm} (21)

Working similarly on (17) we obtain

$$\mathcal{H}_{P,F}(x) \triangleq z^T\left(\left[\tilde{\Phi}(x)\right]^T + F^\tau \Gamma(x)B\right)^T M_x^\tau + M_z\tilde{\Phi}(x)P + B\Gamma(x)F)z < 0, \quad \forall x \not\in B(\tilde{\nu})$$  \hspace{1cm} (22)

The above transformations play a crucial role in our approach: instead of attempting to solve the non-convex optimization problem (18), we solve a relaxed version – which is convex – that involves the functions $\mathcal{F}_{P,F,Q}$ and $\mathcal{H}_{P,F}$:

$$\min \|\mathcal{F}_{P,F,Q}(x)\|^2 + \gamma$$  \hspace{1cm} (23)

s.t.

$$\epsilon_1 I \preceq P \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \bar{Q}$$

$$\mathcal{H}_{P,F}(x) \leq -\gamma, \quad \gamma > c > 0, \forall x \not\in B(\tilde{\nu})$$

where $\epsilon_i, i = 1, 2, 3$ are some positive design constants (with $\epsilon_2 > \epsilon_1$). In the next section, the relationship between the optimization problems (18) and (23) is investigated, and it is shown how the two problems can be made close by opportune tuning of the design parameters. Besides, even for the sake of simplicity, scalar design parameters $\epsilon_i, i = 1, 2, 3$ are considered, all the mathematical results are replicable by considering $P \preceq \bar{P}, \bar{Q} \preceq \bar{Q}$, with $\bar{P}, \bar{Q}$, $\bar{Q}$ matrices of appropriate dimension.

Despite the fact that the optimization problem (23) is a convex problem, its solution requires discretization of the

\footnote{Note that throughout this paper the notation $B(\cdot)$ is used in a similar way as the notation $\mathcal{O}(\cdot)$: $B(a)$ is used to denote a ball of radius proportional to $a$ and not a ball of radius $a$.}
state-space as it is an infinite-dimensional, state-dependent problem. Fortunately, as shown in Sect. V, due to the particular form of (23), the number of discretization points does not have to be as large as it would be required in typical state-dependent optimization problems: as shown in Theorem 1 presented below, the number of discretization points can be as few as the total number of free variables in the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{F}$.

So, instead of (23), it is possible to solve the convex problem

$$\min \sum_{i=1}^{N} \|F_{\mathbf{P}, \mathbf{Q}}(x_i)\|^2 + \gamma$$

subject to

$$\epsilon_1 I \preceq \bar{P} \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq \bar{Q}$$

$$\mathcal{H}_{\mathbf{P}, \mathbf{P}}(x_i) \leq -\gamma, \quad \gamma > 0, \forall x \notin B(\nu)$$

where $N$ is any integer satisfying $N \geq N$ and $N$ denotes the total number of free variables of the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{F}$.

The stability properties of the proposed approach will be mathematically shown in Sect. V. We conclude this section by demonstrating how the proposed method performs on the inverted pendulum example. The ConvCD algorithm (24) is run with $N = 300$, $\epsilon_1 = 1$, $\epsilon_2 = 20$, $\epsilon_3 = 1$, $L = 5$ and the resulting controller is compared to three other controllers: the first one is the controller resulting from the local linearization of the pendulum dynamics around the equilibrium, via the Taylor-MacLaurin approximation $\sin(\chi_1) = \chi_1$, and the solution of the LQ problem associated to the resulting linear system that minimize the performance objective (3). We call this controller “local-linearizing controller”. The second controller is the one resulting from the optimization approach described in [20]. The third controller is the one, in the same form as in (15), which minimizes the cost (3), subject to the actual plant (1)-(2) (and not the approximated one). This problem is solved by numerically optimizing, via the fmincon routine of Matlab, the trajectory from an initial condition to the origin. The resulting controller has been called “simulation-optimized controller”. Also, two more controllers, resulting from the ConvCD algorithm after the approximation of the sin function with $L = 1$ and $L = 3$ mixing functions, has been considered. The approximation procedure is similar to the case with $L = 5$ and, for the sake of brevity, it is not exposed in details.

The six controllers are compared by starting from a common initial condition $x_0$, and by calculating the performance objective (3). The results are shown in Table I. It can be seen how the proposed approach achieves better results than the local-linearizing controller (for $L = 3$ and $L = 5$) and a performance which is comparable to the approach exposed in [20] (for $L = 5$). Besides, the ConvCD approach manages to find a performance which is close to one achieved by the simulation-optimized controller. The performance of the simulation-optimized controller is the best one since the optimization procedure is not based on an approximate model of the plant. Also note how the ConvCD performance increases as the number of mixing functions increases. As a final comment, we found via simulations an estimate of the region of attraction for semiglobal stability. The region of attraction has been estimated using an ellipsoid: call $\mathcal{X}_0$ such an estimate: it is observed that the size of $\mathcal{X}_0$ increases by increasing $L$. In particular we have $\mathcal{X}_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2/2 + x_2^2/2 \leq 1\}$ for $L = 1$, $\mathcal{X}_0 = \{x_1^2/2.25^2 + x_2^2/4.5^2 + u^2 \leq 1\}$ for $L = 3$, and $\mathcal{X}_0 = \{x_1^2/4^2 + x_2^2/9.75^2 + u^2 \leq 1\}$ for $L = 5$. These estimates are less conservative than the estimates obtained using the technique described in the Appendix, which are not reported for the sake of brevity.

### III. General Problem Formulation

In this section, we consider the problem of designing a state feedback controller for a general nonlinear system which assumes the following dynamics

$$\dot{x} = F(x, u)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ denote the vectors of system states and control inputs, respectively and $F$ is a nonlinear vector function (assumed to be continuous). The problem at hand is to construct a state-feedback controller

$$u = k^*(x)$$

which makes the zero equilibrium of the closed-loop system stable and moreover, solves the following constrained optimal control problem:

$$\min_u J = \int_0^\infty \bar{\Pi}(x, u) ds$$

subject to

$$\dot{x} = F(x, u)$$

$$u_{\min} \leq u \leq u_{\max}$$

$$\mathcal{C}(x, u) \leq 0$$

where $\bar{\Pi}$ is a bounded-from-below, continuous function of its arguments; $u_{\min}, u_{\max}$ denote the vectors of minimum and maximum, respectively, allowable control signals; $\mathcal{C} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a smooth convex nonlinear vector function. Inequality (28) can represent a large class of constraints met in practical applications. Without loss of generality, we will assume that the minimum of $\bar{\Pi}(x, u)$ is attained at $\bar{\Pi}(0, 0)$ and that all constraints are satisfied for $(x, u) = (0, 0)$.

Please also note that inequality (28) includes as a special case the limited control constraint (27). However, due to the special treatment of limited control constraints in the next section, we include (27) as a different set of constraints than those of (28).

In order to have a well-posed problem we will assume that the controller solving the optimal control problem (26)-(28) provides closed-loop stability, i.e., we will assume that

(A1) Let $u^* = k(\chi)$ be the controller that solves the optimal control problem (26)-(28). Then, for all admissible initial states $x(0)$, the closed-loop system (25) under the feedback $u^* = k(\chi)$ is stable.
IV. TRANSFORMATIONS & APPROXIMATIONS

A. Adding a – nonlinear – integrator

The first step in the proposed approach is to impose special transformations that render the system dynamics, the constraints and the objective criterion in an appropriate format that is convenient for our developments. To do so, we define a new fictitious “control” input $v$ that is calculated according to

$$\dot{v} = v, \quad u = S(\bar{v}) \quad (29)$$

with $S(\cdot)$ being a smooth and invertible function such that

$$u_{\min} \leq S(\bar{v}) \leq u_{\max} \quad (30)$$

It is worth noticing that by adding the integrator (29) and introducing the function $S$ satisfying (30), the problem of designing the control vector $u$ so that constraint (27) is satisfied is transformed into the problem of designing $v$ that does not need to satisfy a boundedness constraint like (27); while $v$ can take “arbitrary” values, the actual control vector $u$ is restricted – due to the use of function $S$ – to satisfy (27).

We can rewrite system (25) and the constraints (28) as follows:

$$\dot{x} = f(x) + Bv \quad (31)$$

$$C(x) \leq 0 \quad (32)$$

where $\chi = [\chi^T \; \bar{u}^T]^T$, and

$$f(x) = \begin{bmatrix} F(\chi, S(\bar{u})) \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \; I \end{bmatrix}, \quad C(x) = \bar{C}(\chi, S(\bar{u}))$$

B. Approximations using Mixing Signals

The second step in the proposed design is to approximate the transformed system dynamics (31) as well as the objective criterion using smooth mixing signals. More precisely, we let $\beta_i, i = 1, \ldots, L$ denote a set of smooth mixing signals that satisfy the following properties:

$$\beta_i(x) \in [0, 1], \quad \sum_{i=1}^{L} \beta_i(x) = 1, \forall x \quad (33)$$

Remark 1: The mixing functions described by (33) include a large number of descriptors for approximation of nonlinear systems. For example, by using (33), it is possible to describe PWL systems with partition of the state space with polytopic cells [22], [23], as well as many Fuzzy dynamical systems, like the Takagi-Sugeno ones [24]. In the first case the mixing functions will be a smooth version of the state space partition, i.e., if a state cell $S_i$ has the polytopic form $G_i^T x \geq 0$, $i = 1, \ldots, L$, then the corresponding mixing function can be obtained from the sigmoidal function $\beta_i(x) = 1/(1 + e^{-G_i x})$, and subsequent normalization with respect to the other state cells, i.e., $\beta_i(x) = \tilde{\beta}_i(x)/\sum_{j=1}^{L} \tilde{\beta}_j(x)$, $i = 1, \ldots, L$. In such a way, each mixing function is active when the corresponding cell will be active. Note that different mixing functions can be used in place of the sigmoidal one, like the gaussian function used in the tutorial example. In the Takagi-Sugeno case, the mixing functions will have the form of fuzzy membership functions, i.e., $\tilde{\beta}_i(x) = \mu_{S_i}(x)$ where $\mu_{S_i}(x)$ is the membership functions over the fuzzy set $S_i$. Again, normalization is necessary if the membership functions do not sum to one. The interested reader is referred to [22], [24], [25] and references therein for numerical techniques for generating approximations of nonlinear systems using PWL and Fuzzy techniques. PWL and Fuzzy techniques include a large but not exhaustive set of tools for descriptors of nonlinear systems; for example, the proposed method can also handle piecewise nonlinear approximations using Sum of Squares [26].

Remark 2: In the majority of existing AOC designs address the problem of providing a control design which approximates the performance of the optimal controller by increasing the “controller complexity” (e.g., by increasing the number of switching linear controllers). Typically, the resulting AOC design provides an efficient performance if the controller complexity is “large enough” which, in turn, may result in control designs that cannot be practically implemented. The proposed ConvCD approach attacks a different problem: given a controller of a fixed complexity (e.g., given the number of switching linear controllers as well as the strategy for switching among the controllers), ConvCD provides a controller which is as close to the optimal as possible while guaranteeing closed-loop system stability. The practical significance of such an approach is the following: in many practical cases, where a PWL or Fuzzy approximation of the nonlinear systems is available, the complexity of a controller leading to satisfactory performance is also known, without the need to look for an accurate (and computationally expensive) fine approximation of the system.

△

Using the above design considerations for the mixing signals $\beta_i$, we consider the following approximations of the system dynamics and the objective criterion\(^{2}\) as follows:

$$\dot{x} \approx \sum_{i=1}^{L} \beta_i(x) (A_i \tilde{x}(x) + Bv) \quad (35)$$

$$\tilde{\Pi}(x, u) \equiv \tilde{\Pi}(x) \approx \sum_{i=1}^{L} \beta_i(x) (\tilde{x}^T Q_i \tilde{x}) \quad (36)$$

\(^{2}\)In the approximation (35) we omitted, in order to avoid lengthy and complicated formulations, the constant terms. All the results of this paper can be readily extended to the case where an approximation with constant terms is employed. In such a case the the approximation (35) is replaced by the following one

$$\dot{x} \approx \sum_{i=1}^{L} \beta_i(x) (A_i \tilde{x}(x) + a_i + Bv) \quad (34)$$

which can be can be recast as in (35) via the state extension $\tilde{x} = [\tilde{x}(x)^T \; 1]^T$ and

$$A_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}$$
where $A_i, Q_i$ are constant matrices with $Q_i$ being positive semidefinite, and

$$\bar{x}(x) = \begin{bmatrix} \varpi(x) \\ \sigma(x) \end{bmatrix}$$

with $\varpi(x)$ being any smooth (also nonlinear) function of $x$ satisfying

$$\varpi(x) = 0 \iff x = 0$$

and $\sigma(x)$ is defined according to

$$\sigma_i(x) = e^{\alpha C_i(x)} - e^{\alpha C_i(0)} - \eta, \quad i = 1, \ldots, p$$

and $\alpha$ is a large positive constant and $\eta$ is a positive constant. The function $\sigma_i(x)$ in (37) has been chosen in such a way that $\sigma_i(0) = 0$, and, if a particular constraint $C_i(x) \leq 0$ is – or is about to be – violated, then the respective $\sigma_i(x)$ takes a very large value, while it is negligible when the constraint is satisfied.

Remark 3: Contrary to standard PWL approximation [in which case the vector $\bar{x}(x)$ coincides with $x$, as in the inverted pendulum example of Sect. II], the PieceWise NonLinear (PWNL) approximations (35) and (36) might involve the use of the nonlinear vector $\varpi(x)$ and $\sigma(x)$. The use of these nonlinear subvectors within $\bar{x}(x)$ is made for the following reasons:

- **The use of a nonlinear vector $\varpi(x)$ instead of a purely linear one may be proven extremely useful in order to simplify the design for the partitioning of the state space. Typically the introduction of nonlinear terms might help to reduce the number of state-space partitions needed to approximate the nonlinear systems.**

- **The nonlinear term $\sigma(x)$ is included in the control signal so that it includes information on whether a particular constraint is – or is about to be – violated.**

As a final step, the constrained optimization problem (26)-(28) is transformed into an *unconstrained* one, by incorporating the constraints (28) – or, equivalently, the constraints (32) – as penalty functions into the objective function. This is made possible by making use of (37) and by replacing the objective function (26) by the following one

$$J = \int_0^\infty \left( \sum_{i=1}^L \beta_i(x(s)) (\bar{x}(s))^T Q_i \bar{x}(s) \right) ds$$

$$+ \sum_{j=1}^{\dim(C(x))} \sigma_j^2(x(s)) ds$$

By using the fact that $\sum_{i=1}^L \beta_i(x) = 1$ and the definition of $z$ in (8), the augmented objective function $J$ can be written in the following compact form:

$$J = \int_0^\infty (z^T(s)Qz(s)) ds$$

where

$$Q = \begin{bmatrix} Q_1 + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & Q_L + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}$$

C. HJB and Controller Approximations

After applying all transformations presented in the previous section, we have that the optimal state feedback design problem can be cast as an unconstrained optimal control problem of the form

$$\text{minimize } J = \int_0^\infty (z^T(s)Qz(s)) ds$$

subject to

$$\dot{x} = \Phi(x)z(x) + Bv + \nu$$

where $\nu$ stands for the approximation error due to the replacement of the actual system dynamics (31) by (35).

As the proposed approach employs approximation techniques it is not possible to provide a globally stable feedback control design. Instead a semi-globally stable feedback control design can be provided which, given a compact subset $X_0 \subset \mathbb{R}^n$, of admissible initial states constructs a control design that produces closed-loop stable solutions provided the initial state $x(0)$ belongs to $X_0$. Therefore, additionally to (A1) we will assume that

(A2) The initial states $x(0)$ of the system belong to a compact subset $X_0 \subset \mathbb{R}^n$.

Associated to the subset $X_0$ of admissible initial conditions we will define a “sufficiently large” compact subset $X$ which contains $X_0$, such that every trajectory starting inside $X_0$ will be contained in $X$. An estimate of the size of the region of attraction is provided in the proof of the main Theorem on the Appendix.

Remark 4: Note that, in case $x$ lies in the compact subset $X \subset \mathbb{R}^n$, then the amplitude of the term $\nu \equiv \nu(x)$ can become arbitrarily small and is inversely proportional to the number $L$ of mixing signals.

Moreover, in order for the approximators of the optimal cost-to-go function and the respective optimal controller to be able to provide with arbitrary accuracy, the optimal cost-to-go function and the respective optimal controller continuous. In other words, we additionally have to impose the following assumptions.

(A3) The optimal cost-to-go function $V(x)$ and the respective optimal control $v^*$ which correspond to the HJB solution are continuous functions (wrt. $x$).

V. THE CONVCD APPROACH

Following the same mathematical steps exposed after the HJB equation (12), we arrive to the convex optimization problem (23).
The following Lemma, which justifies the choice of (13) as an approximation of the optimal-cost-to-go function is introduced.

Lemma 1: Let (A1)-(A3) hold and assume that $x \in \mathcal{X}$. The optimal-cost-to-go function $V$ can be approximated – with accuracy $O(1/L)$ – using a Sum-of-Squares (SoS) polynomial (13), with $P$ being symmetric and having the block diagonal form of (14) and $P_\alpha$ are $\dim(\bar{x})^2$-dimensional symmetric and positive definite matrices.

Proof: See the Appendix.

Remark 5: Similarly to [17]-[19], the proof of Lemma 1 is based on the observation that the optimal cost-to-go function $V$, being a Control Lyapunov Function (CLF) for the controlled system, can be approximated using a SoS polynomial. However, that while in [17]-[19], the vector $z$ is vector of monomials, whose size increases exponentially while increasing the order of the monomials, in (13) the size of $z$ increases linearly by increasing the number of mixing elements. Due to the use of piecewise approximations, the matrix $P$ in (13) is of block diagonal structure which reduces significantly the computational burden of the ConvCD design as compared to the use of high order monomials in [17]-[19].

Remark 6: At this point we should emphasize that the two optimization problems (18) and (23) are not equivalent, being (23) a convex relaxation of the non-convex problem (18). This is due to the fact that the optimal solution coming from (23) might not satisfy

$$Q = \hat{P}^{-1}\bar{Q}\hat{P}^{-1} \tag{42}$$

Condition (42) is a non-convex constraint which cannot be imposed without destroying the convexity of (23). To solve such a problem, the condition (42) is “approximately” imposed by appropriately re-defining $\hat{x}(x), \bar{Q}, \hat{P}$ and $Q$. Considering for simplicity the case where no constraints (28) are present, we augment the vector $\hat{x}$ by adding the element $\bar{V}_{\bar{P}}$ as its first component. Then it suffices to choose $Q = \text{diag}(1, \delta, \ldots, \delta)$ where $\delta$ is a small positive design constant and restrict the matrices $\hat{P}$ and $\bar{Q}$ to admit following block diagonal form

$$\bar{P} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \bar{P}_1 & 0 & \cdots & 0 \\
0 & 0 & \bar{P}_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \bar{P}_L
\end{bmatrix}, \quad \bar{Q} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \bar{Q}_1 & 0 & \cdots & 0 \\
0 & 0 & \bar{Q}_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \bar{Q}_L
\end{bmatrix}$$

where $\bar{P}_i, \bar{Q}_i$ are symmetric positive definite matrices constrained to satisfy $\epsilon_1 I \preceq \bar{P}_i \preceq \epsilon_2 I$, $\epsilon_3 I \preceq \bar{Q}_i \preceq \delta I$. Under such a choice it is not difficult to see that $Q = \bar{P}^{-1}\bar{Q}\bar{P}^{-1} + O(\delta)$ and thus condition (42) approximately holds. Therefore, by imposing the above described redefinitions, the two optimization problems (18) and (23) become approximately equivalent, i.e., the smaller is $\delta$, the closer are the solutions of the two problems. Similar – but lengthier – transformations can be imposed in the case where constraints of the form (28) are present.

In the sequel and in order to avoid lengthy complications, we will assume that condition (42) holds for a particular choice of the design constants $\epsilon_i$ and the set-up described so far i.e., without needing to redefine $\hat{x}(x), \bar{Q}, \hat{P}$ and $Q$ as described above. All the arguments of the paper can be straightforwardly extended in the case where (42) does not hold, in which case redesign/redefinition of $\hat{x}(x), \bar{Q}, \hat{P}$ and $Q$ is required.

Table II presents the proposed procedure for solving the optimization problem (23) and constructing the proposed control design scheme. Despite the fact that the optimization problem (23) is a convex problem, its solution requires discretization of the state-space as it is an infinite-dimensional, state-dependent problem. Fortunately, due to the particular form of (23), the number of discretization points does not have to be as large as it is required in typical state-dependent optimization problems: as it will be seen in Theorem 1 presented below, the number of discretization points can be as few as the total number of free variables in the matrices $\hat{P}, \bar{Q}, F$. This is contrary to alternative AOC approaches that typically require a dense grid covering the state-space, leading to a large number of discretization points.

Remark 7: The matrices $\hat{P}, \bar{Q}$ should be positive definite. As a result any approach that attempts to solve (43) should constrain these matrices to be positive definite; SDP methods should be engaged towards such a purpose. Please note that due to the special structure of the matrices $\bar{Q}, \hat{P}$ given in (39) and (14), respectively, the SDP constraints

$$\epsilon_1 I \preceq \hat{P}, \quad \epsilon_3 \bar{Q} \preceq \bar{Q}$$

in (43), can be broken down into $L$ independent smaller constraints, thus decreasing the computational load. Finally, Schur complement techniques – see e.g. [27] – may be used to transform the SDP subject to second-order cone constraints (43) into a SDP subject to Linear Matrix Inequality (LMI) constraints.

The next theorem establishes the stability properties of the overall scheme presented in Table II.

Theorem 1: Fix the number $L$ of mixing signals and the constants $\epsilon_i, i = 1, 2, 3$, let $P, G$ be constructed according to the design procedure of Table II and let (A1)-(A3) hold. Let also $M$ be any positive integer satisfying $M \geq N$, where $N$ – see also Table II – denotes the number of free variables of the matrices $\hat{P}, \bar{Q}, F$. Select1 randomly $M$ points $x^{(j)} \in \mathcal{X}$ and let

$$\hat{z}^{[j]} = z(x^{(j)})$$

Finally, suppose that the positive design constants $\epsilon_i$ are chosen so that

$$\epsilon_1 I \preceq \hat{P}^{*-1} \preceq \epsilon_2 I, \quad \epsilon_3 \bar{Q} \preceq \bar{Q}^{*-1}$$

where $G^*, P^*$ denote the optimal solutions to the optimization problem (18).

Then, the following statements hold:

(a) Let $C_1 > 0$ be the smallest positive constant such that, for all $j \in \{1, \ldots, M\}$,

$$\|\hat{z}^{[j]}\| > C_1 \Rightarrow \mathcal{L}_{P, C_1}(x^{[j]}) < 0 \tag{46}$$

Note that although it is possible for $\mathcal{N}$ and $x^{[j]}$ of Table II to coincide with $M$ and $x^{(j)}$, respectively, it is advisable that they do not coincide so that the performance of the proposed control design is evaluated at different state points than the ones used to construct the ConvCD solution.
Table II: The ConvCD Approach

Step 1. Calculate the matrices $P, Q, F$ as follows: Let $N$ denote the total number of free variables of these matrices. Select randomly $N$ points $x^{[i]} \in X$, where $N$ is any integer satisfying $N \geq N$ and solve the following convex optimization problem:

$$\min_{i=1}^{N} \|F_{P,F,Q}(x^{[i]})\|^2 + \gamma$$

subject to

$$\epsilon_1 I \preceq P \preceq \epsilon_2 I, \quad \epsilon_3 Q \preceq Q$$

where $\epsilon_1, \epsilon_2, \epsilon_3$ are user-defined positive constants, and $F_{P,F,Q}(x^{[i]})$ and $H_{P,F}(x^{[i]})$ as defined respectively in (19) and (22).

Step 2. By using the solution of the above optimization problem, we can extract the estimates of the matrices $P, Q$ in (18) according to

$$P = P^{-1}, \quad G = FP^{-1}$$

Step 3. The proposed control scheme is the controller given by (15) and (29) by setting $G$ equal to $G$.

then, the closed-loop system is stable and its solutions converge to the subset

$$D = \left\{ x : \|z(x)\| \leq C_1 + \mathcal{O}\left( \frac{1}{M} \right) \right\}$$

Moreover, the closed-loop solutions of the system satisfy

$$\|x(t) - x_{opt}(t)\| \leq C_2 + \mathcal{O}\left( \frac{1}{M} \right)$$

where $x_{opt}$ denotes the closed-loop solutions of the system under the optimal control $v^*$ and $C_2$ is a nonnegative constant satisfying

$$C_2 = \mathcal{O}\left( g_{P,F}(x^{[j]}) \right)$$

(b) For each $C_1 > 0, C_2 \geq 0$, there exists a lower bound on the approximators’ size $L$ so that (46) and (48) hold for all choices of approximators’ size $L$ satisfying $L \geq L$.

Proof: See the Appendix

Several remarks are in order:

- The optimization problem (23) is a convex one: the optimization criterion comprises a quadratic function with respect to the decision variables $P, F, Q$ while all of the constraints are Semi-Definite constraints (and thus convex).
- Most importantly, according to Theorem 1, the solution to the optimization problem (23) is able to provide an efficient control design even in cases where the approximation error is not negligible. Theorem 1 provides an easy-to-calculate formula – see relation (46) – to check whether a particular choice for the approximators size $L$ provides the required controller efficiency. In other words, even in cases where the particular choice for $L$ is far from providing a close-to-the-optimal performance (i.e., a significantly larger $L$ – and thus a significantly more complicated controller – is required to get a close-to-optimal performance), the proposed scheme provides a control design that is efficient.

Furthermore to Theorem 1 and by using the same arguments as in [17] it can be seen that in case (46) holds, then the solutions of the overall system satisfy the following inequality:

$$\|z(t)\| \leq \frac{\alpha_1 \exp^{-\alpha_2 t}}{\epsilon_1} \|z(0)\| + \alpha_3$$

where $\epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$ satisfy (43) and (44)

What is important about (49) is that the design constants $\epsilon_i$ in the optimization problem (56) can serve as tuning/design parameters in a similar fashion as e.g., the LQ matrices in Linear-Quadratic control design applications: (49) can be used to evaluate the effects and trade-offs of different choices for $\epsilon_i$ on the overshoot, convergence and steady-state closed-loop performance and thus it can provide a guide on how to choose $\epsilon_i$ so that the desired performance is obtained.

Remark 8: We close this section by emphasizing the fact that, in the case where $\hat{x}(x) \equiv x$ and no input/state constraints are given, because of the block diagonal structure of the Lyapunov matrix (13)-(14), the computational complexity of the control problem is comparable with the solution of $L$ Riccati equations, i.e., the computational requirements of the control scheme resulting using the proposed approach increases linearly with the complexity of the controller.

VI. NUMERICAL EXAMPLE: FREEWAY MODEL

In this section, we demonstrate how the methodology we presented can be applied in a second, more realistic example: congestion reduction on a freeway. Traffic on a freeway can be described by a nonlinear system, composed of many interconnected subsystems, namely the sections by which the freeway is divided into, see Fig. 2. One of the control strategies proposed in the literature to control congestion on a freeway is the use of variable speed limits across the sections [28], [29]. The dynamics of a freeway with $S$ sections are described by a nonlinear dynamical system, whose exact description can be found in [28], [29], and which takes the form

$$\dot{\rho} = G(\rho, \nu)$$

where $\rho = [\rho_1, \ldots, \rho_S]^T$ are the mean densities across the sections and $\nu = [\nu_1, \ldots, \nu_S]^T$ the speed limits in each sections.
The system dynamics (50) are subject to the constraints:

\[ 0 \leq \rho_i \leq \rho_{jam}, \quad i = 1, \ldots, S \]
\[ v_{min} \leq v_i \leq v_f, \quad i = 1, \ldots, S \]

with \( v_f \) the free flow speed and \( v_{min} \) the minimum speed limit. The system (50) has a fixed point equilibrium \( \rho_i^{eq} = \frac{q_i}{v_f}, \quad i = 1, \ldots, S \) and \( u_i^{eq} = v_f, \quad i = 1, \ldots, S \), which represents the equilibrium in the free flow condition, to which we would like to drive the system. So we define \( \chi_i = \rho_i - \rho_i^{eq}, \quad i = 1, \ldots, S \) and \( u_i = v_i - v_i^{eq}, \quad i = 1, \ldots, S \), and we apply the transformations to obtain a system in the form of (31).

\[
\begin{array}{c}
\text{section } j & \rightarrow \rho_j, v_j \\
\text{section } i & \rightarrow \rho_i, v_i \\
\text{section } N & \rightarrow \rho_N, v_N
\end{array}
\]

For the control problem at hand we consider a freeway model with \( S = 4 \) sections, and \( \rho_{jam} = 0.75 \text{[veh/m]} \), \( v_f = 29.17 \text{[m/s]} \), \( v_{min} = 13.89 \text{[m/s]} \). The control objective is to choose \( u_i, \quad i = 1, 2, 3, 4 \), so as to optimize the criterion

\[
J = \int_0^\infty \left[ \sum_{i=1}^{4} \chi_i^2(\tau) + 10^{-4} \sum_{i=1}^{4} u_i^2(\tau) \right] d\tau
\]

Such a criterion describe the desired to operate close to the free flow conditions while keeping the control action small. The above criterion along with the constraints (51) and (52) were transformed into the unconstrained objective criterion \( J \) using the approach detailed in section III.

The ConvCD algorithm described in Table II was implemented with \( e_1 = 1 \), \( e_2 = 2 \), \( e_3 = 10^{-2} \) and \( N = 2000 \) discretization points. Two types of synthesis have been made, one employing just one mixing function, \( L = 1 \) (\( \beta_1 \equiv 1 \)), and a second one employing 2 mixing functions, \( L = 2 \). In the second case, the two mixing functions have been chosen in the following way. Consider the pre-normalized weights, \( i = 1, 2 \)

\[
\tilde{\beta}_i(\chi) = e^{-\frac{\chi_{01}^2}{\chi_{02}^2} (\chi_i^2 - \chi_{0i}^2)}
\]

where \( \chi_{01} = 0, \chi_{02} = 0.6 \), and \( \sigma_\chi = 0.5 \). These weights represent the freeflow freeway condition and the congested freeway condition. The mixing signals \( \beta(\chi) \) are generated by normalizing \( \tilde{\beta} = [\beta_1 \beta_2] \), i.e., \( \beta_i(\chi) = \tilde{\beta}_i(\chi) / \sum_{j=1}^{2} \tilde{\beta}_j(\chi) \), \( i = 1, 2 \). Different mixing functions can be considered, but the following choice has been verified to approximate in a satisfactory manner the 4-section freeway.

Both controllers found from the ConvCD methodology with \( L = 1 \) and \( L = 2 \) have been tested in simulations under a congested scenario, and compared with the not controlled (or open loop) strategy, i.e., when \( v_i(t) \equiv v_f, \forall t \). The objective function (53) has been calculated, in order to evaluate the improvement respect to the case with no control action. The results are summarized in Table III. Both controllers coming from the ConvCD procedure improves the performance index coming from the open loop strategy. The controller employing 2 mixing functions achieves a better performance than the controller with 1 mixing function.

In Fig. 3, 4, 5 the four mean densities across the section, as well as the four variable speed limits are shown, respectively for the open loop case, for the ConvCD controller with \( L = 1 \), and for the ConvCD controller with \( L = 2 \). It can be seen that, as soon as congestion appears in the fourth section, the two ConvCD controllers decrease the speed limits in the previous sections, in order to allow less cars to enter into the congested zone (Fig. 4(b) and 5(b)): this strategy allows a decrease in the propagation of the congestion wave to the previous sections of the freeway. An estimate of the region of attraction for semiglobal stability for the two controllers is: \( \bar{X}_0 = \{[\rho^* \nu^*] \tau : 0 \leq \rho_i \leq \rho_{jam}/4.5, v_{min} \leq v_i \leq v_f, i = 1, \ldots, 4 \} \) for \( L = 1 \) and \( \bar{X}_0 = \{[\rho^* \nu^*] \tau : 0 \leq \rho_i \leq \rho_{jam}/3.25, v_{min} \leq v_i \leq v_f, i = 1, \ldots, 4 \} \) for \( L = 2 \), which are acceptably inside the practical operating range of the freeway system.

**VII. Conclusions & Future Research**

A control design methodology for Approximately Optimal Control (AOC) of nonlinear systems has been presented. The proposed design, called ConvCD, transforms the problem of constructing an AOC into a convex problem, involving the solution of a Semi-Definite Programming (SDP) problem. The ConvCD methodology, is scalable in the sense that it involves SDP constraints of the same order as the system dimension. The stability properties of the proposed approach have been analyzed: the proposed scheme guarantees semiglobal stability of the nonlinear closed-loop system. Besides,
given a user-defined optimality criterion, the approximately optimal controller coming from ConvCD can approximate with arbitrary accuracy the performance of the actual optimal controller. Two nonlinear numerical examples have been used to show the effectiveness of the method. As ConvCD assumes perfect knowledge of the system dynamics it may become inefficient in cases of system uncertainties or variations or, even worse, in cases where minor or major faults or anomalies affect the system dynamics. In order for ConvCD to be practically efficient, an adaptive self-tuning/re-design tool is required that will take care of all the above-mentioned factors that may affect the efficiency of ConvCD. This will be a topic of future research.

ACKNOWLEDGEMENTS

The research leading to these results has been partially funded by the European Commission FP7-ICT-2013.3.4, Advanced computing, embedded and control systems, under contract #611538 (LOCAL4GLOBAL).

APPENDIX

PROOF OF LEMMA 1

It is quite straightforward to show similarly to [17]-[19] that – due to (A1) – $V(x)$ is a positive definite function [in fact $V(x)$ is nothing but a Control Lyapunov Function (CLF)]. Since $V(x)$ is positive definite and $\mathcal{X}$ is a compact subset, we have that there exists a positive constant $p^*$ such that the following function

$$W(x) = V(x) - p \sum_{j=1}^{\dim(C(x))} \sigma_j^2(x)$$

is also positive definite for all $x \in \mathcal{X}$ and all $p \in (0, p^*)$ [please note that by assumption, all constraints are satisfied for $(\chi, u) = (0, 0)$ and thus $\sigma_j^2(0) = 0$]. It can be seen – using similar arguments as in [17], [18], [19] and standard function approximation arguments – that the function $W(x)$ can be approximated – with accuracy $O(1/L)$ – by the function $\sum_{i=1}^L \beta_i(x)\varpi(x)^T \mathbf{P}_i \varpi(x)$ with $\mathbf{P}_i$ being symmetric and positive definite. The proof of (13) is established by setting $\mathbf{P}_i = \mathbf{P}_i + \begin{bmatrix} 0 & 0 \\ 0 & p \mathbf{I} \end{bmatrix}$.

PROOF OF THEOREM 1

Consider the following two optimization problems:

$$\min \left\| \mathcal{F}_{\mathbf{P}, \mathbf{F}, \mathbf{Q} + \epsilon_3 \mathbf{Q}}(x) \right\|^2 + \gamma$$

s.t.

$$\epsilon_1 I \preceq \tilde{\mathbf{P}} \preceq \epsilon_2 I, \quad 0 \preceq \tilde{\mathbf{Q}}$$

$$\mathcal{H}_{\mathbf{P}, \mathbf{F}}(x) \leq -\gamma, \quad \gamma > c > 0, \forall x \notin \mathcal{B}(\bar{\nu})$$

and

$$\min \sum_{i=1}^N \left\| \mathcal{F}_{\mathbf{P}, \mathbf{F}, \mathbf{Q} + \epsilon_3 \mathbf{Q}}(x^{[i]}) \right\|^2$$

s.t.

$$\epsilon_1 I \preceq \tilde{\mathbf{P}} \preceq \epsilon_2 I, \quad 0 \preceq \tilde{\mathbf{Q}}$$

$$\mathcal{H}_{\mathbf{P}, \mathbf{F}}(x^{[i]}) \leq -\gamma, \quad \gamma > c > 0, \forall x \notin \mathcal{B}(\bar{\nu})$$

The optimization problems (55) and (56) are equivalent to the optimization problems (23) and (43), respectively. From now on, we will consider instead of the optimization problems (23) and (43), their equivalent ones (55) and (56).

Let $\vartheta$ denote a vector which contains all non-zero entries of the matrices $\mathbf{P}, \mathbf{Q}, \mathbf{F}$: as the first matrices symmetric block-diagonal, the dimension of the vector $\vartheta$ is smaller than the number of non-zero entries of $\mathbf{P}, \mathbf{Q}, \mathbf{F}$. Similarly, let $\theta$ denote a vector which contains all non-zero entries of the matrices $\mathbf{P}$ and $\mathbf{G}$. Let also $\vartheta^*$ and $\theta^*$ denote the global optimizers of the infinite-dimensional problems (55) and (18), respectively. Finally, let $T$ denote the one-to-one transformation from $\theta$ to $\vartheta$ defined according to $\mathbf{P} = \mathbf{P}^{-1}, \mathbf{G} = \mathbf{F}\mathbf{P}^{-1}, \mathbf{Q} = \mathbf{P}^{-1}\mathbf{Q}\mathbf{P}^{-1}$. Then, if (45) holds, the following relation is true

$$\vartheta^* = T(\vartheta^*) + \mathcal{O}(\bar{\nu})$$

Using the definition of $\vartheta$ we have that

$$\mathcal{F}_{\mathbf{P}, \mathbf{F}, \mathbf{Q} + \epsilon_3 \mathbf{Q}}(x) = \vartheta^T \phi(x) + \epsilon_3 z(x)^T \mathbf{Q} z(x)$$

for some appropriately defined function $\phi$. Moreover, from (19) we have that

$$\vartheta^* \phi(x) + \epsilon_3 z(x)^T \mathbf{Q} z(x) = \mathcal{O}(\bar{\nu})$$

Let

$$\Phi = \sum_{i=1}^N \phi^T(x^{[i]})\phi(x^{[i]})$$

and

$$Z = -\epsilon_3 \sum_{i=1}^N \phi(x^{[i]})z^T(x^{[i]})\mathbf{Q} z(x^{[i]})$$

Table III: Freeway control results

<table>
<thead>
<tr>
<th>$J$</th>
<th>No control</th>
<th>ConvCD (L=1)</th>
<th>Improv.</th>
<th>ConvCD (L=2)</th>
<th>Improv.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1318.7$</td>
<td>$1215.5$</td>
<td>$7.8%$</td>
<td>$1141.6$</td>
<td>$13.4%$</td>
<td></td>
</tr>
</tbody>
</table>
Then, by using (59) we directly obtain

$$\Phi \hat{\vartheta} = Z + O(\nu)$$  \hfill (60)

We will show that the matrix $\Phi$ is full-rank with probability 1. The proof of the last claim can be established using similar arguments as in [30], [18]; if $\Phi$ is not full-rank then there exists a constant non-zero vector $b$ such that

$$b^\top \Phi(x[i]) = 0, \forall i \in \{1, \ldots, N\}$$  \hfill (61)

which is equivalent to a set of nonlinear system of equations in $x[i]$. The set of solutions to these system of equations has zero Lebesgue measure, see e.g. [30], [18], and as a result under a random choice for $x[i]$ the probability that (61) holds is zero. In other words, a random choice for $x[i]$ guarantees that $\Phi$ is full-rank with probability 1.

Let $\hat{\vartheta}$ denote the solution of the ConvCD optimization problem (56) and

$$J(\hat{\vartheta}) = ||\Phi \hat{\vartheta} - Z||^2$$

Since $\hat{\vartheta}^\ast$ is a feasible solution to the optimization problem (56) and by using (60), we have that

$$J(\hat{\vartheta}) \leq J(\vartheta^\ast) = O(\nu)$$  \hfill (62)

As a result, since $J(\cdot)$ is a quadratic function, we have that

$$J(\hat{\vartheta}) = J(\vartheta^\ast) + 2(\vartheta^\ast - \hat{\vartheta})^\top \Phi^\top (\Phi \vartheta^\ast - Z) + (\vartheta^\ast - \hat{\vartheta})^\top \Phi^\top \Phi (\vartheta^\ast - \hat{\vartheta})$$  \hfill (63)

Combining (62) and (63) we obtain

$$0 \geq 2(\vartheta^\ast - \hat{\vartheta})^\top \Phi^\top (\Phi \vartheta^\ast - Z) + (\vartheta^\ast - \hat{\vartheta})^\top \Phi^\top \Phi (\vartheta^\ast - \hat{\vartheta})$$

Using the above inequality together with the facts that (a) $\Phi$ is full-rank and thus $\Phi^\top \Phi > 0$ and (b) $||\Phi \vartheta^\ast - Z|| = O(\nu)$ we finally obtain that

$$\hat{\vartheta} = \vartheta^\ast + O(\nu)$$  \hfill (64)

that is, the solution of (56) is close close to the solution of (55), apart from a term of the order of magnitude of the approximation error $\nu$. The above equation in combination with (57) establishes that

$$\vartheta^\ast = T(\hat{\vartheta}) + O(\nu)$$  \hfill (65)

Combining (65) with (16) we finally obtain that

$$\mathcal{G}(x) = O(\nu)$$  \hfill (66)

where $\hat{\mathcal{P}}$, $\hat{\mathcal{G}}$ are generated by transforming the solution of the ConvCD optimization algorithm (43) [or, its equivalent, (56)].

Besides, from (45), (58) and (66), by defining $\tilde{V}(x) = z^\top \hat{\mathcal{P}} z$, it is possible to verify that

$$1/\epsilon_2 ||z||^2 \leq \tilde{V}(x) \leq 1/\epsilon_1 ||z||^2$$  \hfill (67)

$$\tilde{V}(x) \leq -1/\epsilon_3 Q ||z||^2 + O(\nu)$$  \hfill (68)

Now, define $k_1 = 1/\epsilon_2$, $k_2 = 1/\epsilon_1$, $k_3 = \lambda_{\min}(Q)/\epsilon_3$, where $\lambda_{\min}(Q)$ is the smallest eigenvalue of the matrix $Q$, we obtain [31]

$$\tilde{V}(x(t)) \leq \exp\left(-\frac{k_3}{k_2}t\right)\tilde{V}(x(0)) + O(\nu)$$  \hfill (69)

$$||z(t)|| \leq \sqrt{\frac{k_2}{k_1}} \exp\left(-\frac{k_3}{2k_2}t\right)||z(0)|| + O(\nu)$$  \hfill (70)

So we proved that for $N$ randomly selected points, Eq. (66) is verified and, moreover, the closed-loop is stable (for $L$ sufficiently large) and converges to a set centered at zero and having radius proportional to $O(1/L)$.

In order to estimate the region of attraction for semiglobal stability, we use the uniformly boundedness results from [31]. Suppose that the sets $\Omega_{\epsilon} = \{\tilde{V}(x(t)) \leq \tilde{\epsilon}\}$, $\Lambda = \{\tilde{\epsilon} < \tilde{V}(x(t)) \leq \tilde{\epsilon}\}$, with $\tilde{\epsilon} > 0$, are compact. Supposing (68) is valid $\forall x \in \Lambda$, and defining

$$k = \min_{x \in \Lambda} \{\tilde{V}(x)\} = O(\nu)$$

we have

$$B_{\mu} = \{x : ||x|| \leq \mu\}, B_r = \{x : ||x|| \leq r\}$$  \hfill (72)

and

$$X_0 = \{x : ||x|| \leq \tilde{\mu}\}, \mathcal{X} = \{x : ||x|| \leq \tilde{r}\}$$  \hfill (74)

Moreover, from the HJB equation we have that the optimal cost-to-go $V$ time-derivative under the optimal state-feedback $v^\ast = k(x)$ satisfies

$$\hat{V} = \frac{\partial V}{\partial x}(f(x) + Bk(x) + \nu) = -\Pi(x) = -z^\top Qz + O(\nu)$$  \hfill (75)

The above equation together with (68) and the boundedness of the closed-loop solutions of the proposed controller establish that $\tilde{V}(t) = V(t) + O(\nu)$, $\tilde{V}(t) = \tilde{V}(t) + O(\nu)$ and therefore $x(t) = x_{opt}(t) + O(\nu)$. The rest of the proof can be established using standard function approximation arguments (i.e., that for a $O(1/L)$ term and any $\epsilon > 0$ we have that there exist a $L$ such that the magnitude of the term $O(1/L)$ is smaller than $\epsilon$ for all $L > \tilde{L}$) as well as the fact that for any function $f(x)$ its maximum amplitude can be estimated to be equal to the maximum value $|f(x)|$ along randomly selected points $x[i], i = 1, \ldots, M$, with accuracy that is proportional to the number $M$ of random points.

References


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